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An algebraic method for finding a series of exact solutions to integrable and nonintegrable nonlinear evolution equations

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Abstract
An algebraic method is devised to uniformly construct a series of exact solutions for general integrable and nonintegrable nonlinear evolution equations. Compared with most existing tanh methods, the Jacobi function expansion method or other sophisticated methods, the proposed method not only gives new and more general solutions, but also provides a guideline to classify the various types of the solutions according to the values of some parameters. The solutions obtained in this paper include (a) polynomial solutions, (b) exponential solutions, (c) rational solutions, (d) triangular periodic wave solutions, (e) hyperbolic and solitary wave solutions and (f) Jacobi and Weierstrass doubly periodic wave solutions. The efficiency of the method can be demonstrated on a large variety of nonlinear equations such as those considered in this paper, new $(2+1)$-dimensional Calogero–KdV equation, $(3+1)$-dimensional Jimbo–Miwa equation, symmetric regular long wave equation, Drinfel'd–Sokolov–Wilson equation, $(2+1)$-dimensional generalized dispersive long wave equation, double sine-Gordon equation, Calogero–Degasperis–Fokas equation and coupled Schrödinger–Boussinesq equation. In addition, the links among our proposed method, the tanh method, the extended method and the Jacobi function expansion method are also clarified generally.

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1. Introduction

The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solution, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. In the past decades, there has been significant progression in the development of these methods such as the inverse scattering method [1], Darboux transformation [2–7], Hirota bilinear method [8, 9], Lie group method [10, 11], algebro-geometric method [12–15] and tanh method [16–18]. Among those, the tanh method provides a straightforward and effective algorithm to obtain such particular solutions for a large nonlinear equation. Based on the fact that solitary wave solutions are essentially of localized nature, one can write the solitary wave solutions of a nonlinear equation as the polynomials of hyperbolic functions and change it into a nonlinear system of algebraic equations. In recent years, much research work has been concentrated on the various extensions and applications of the tanh method [17–23]. The basic purpose of these papers is to simplify the routine calculation of the method and to find more general exact solutions. Parkes and Duffy mentioned the difficulty of using the tanh method by hand for anything but simple partial differential equations. Therefore, they automated to some degree the tanh method using symbolic computation software *Mathematica* [18]. We presented a generalized tanh method for obtaining multiple travelling wave solutions [20, 21]. The key idea is to use the solution of a Riccati equation to replace the tanh function in expression (1.3). Recently, a new algorithm based on Wu's method and computer software *Maple* was presented by Li and Yao to automate the tanh method [22, 23].

In this paper, we shall develop an algebraic method with computerized symbolic computation, which greatly exceeds the applicability of the existing tanh, extended tanh methods and Jacobi function expansion method in obtaining a series of exact solutions of nonlinear equations. The obtained solutions may include (a) polynomial solutions, (b) exponential solutions, (c) rational solutions, (d) triangular periodic wave solutions, (e) hyperbolic and solitary wave solutions and (f) Jacobi and Weierstrass doubly periodic wave solutions. We remark here that by applying spectral theory, Weierstrass and theta elliptic functions can be used to find periodic solutions for some equations such as the KdV equation, coupled nonlinear Schrödinger equation etc. But this method is usually applied in the integrable nonlinear evolution equations admitting Lax pairs representation [12–15]. An alternative method is to transform the equation under study to the Weierstrass equation, Jacobi equation, or more generally, to Painlevé-type equations [24–26]. This procedure is in general complicated or impossible, especially for complicated dissipative nonlinear equations and nonlinear coupled systems. Very recently, a Jacobi function expansion method was applied to construct periodic wave solutions for some nonlinear equations. The essential idea of this method is similar to the tanh method by replacing the tanh function with some Jacobi elliptic functions such as $\mathrm{cn}\,\xi$, $\mathrm{sn}\,\xi$ and $\mathrm{dn}\,\xi$ [27, 28]. For example, Jacobi periodic solution in terms of $\mathrm{sn}\,\xi$ may be obtained by applying sn-function expansion. To get Jacobi doubly periodic wave solutions in terms of $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$, many similar repetitious calculations have to be made, and these efforts will be in vain if an equation does not admit these types of solutions at all. The feature of our method proposed here is that, without much extra effort, we circumvent integration to directly get the above series explicit solutions (a)–(f) in a uniform way, which readily covers all results of the tanh method, extended tanh method, Jacobi function expansion method and some other sophisticated methods. Another merit of our method is that it is independent of the integrability of nonlinear equations. Viewed as a special case of partial differential equations, the method readily applies to nonlinear ordinary differential equations.

Our paper is organized as follows. In section 2, the detailed deviation of the proposed method is given. The applications of the proposed method to nonlinear integrable and nonintegrable evolution equations are illustrated in section 3. The conclusion is then given in section 4.

2. The proposed method

Let us first recall how the tanh method works. For a given partial differential equation in $u(x,t)$

$$H(u, u_t, u_x, u_{xx}, \ldots) = 0 \tag{2.1}$$

where for convenience we take $u(x,t) = U(\xi) = U(x+ct)$, which is just the same as for ordinary travelling frame $u(x,t) = U(\xi) = U(x-ct)$ under transformation $c \to -c$. Now, in our travelling frame we may transform the partial differential equation (2.1) into an ordinary differential equation

$$H(U', U'', \ldots) = 0. \tag{2.2}$$

The next crucial step is that the solution sought for is expressed as a polynomial in tanh function, namely,

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i \tanh^i \xi. \tag{2.3}$$

Doing so, we may take advantage of the property that the derivative of $\tanh\xi$ is polynomial in $\tanh\xi$, i.e. $(\tanh\xi)' = 1 - \tanh^2\xi$. The positive integer n is determined by balancing the highest order linear term with the nonlinear terms. By substituting (2.3) into equation (2.2) and setting all coefficients of powers of $\tanh\xi$ to zero, we obtain a system of algebraic equations, from which the parameters a_i and c are explicitly obtained.

Now we outline our method, whose key idea is to take advantage of a first-order ordinary differential equation and use its solutions to replace the tanh function in the expression (2.3). The main steps are given as follows.

Step 1. Reduce partial differential equation (2.1) to the ordinary differential equation (2.2) by considering the wave transformation $u(x,t) = U(\xi)$, $\xi = x + ct$.

Step 2. Expand the solution of equation (2.2) in the form

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i \varphi^i \tag{2.4}$$

where the new variable $\varphi = \varphi(\xi)$ is a solution of the following ordinary differential equation

$$\varphi' = \varepsilon \sqrt{\sum_{j=0}^{r} c_j \varphi^j} \tag{2.5}$$

and $\varepsilon = \pm 1$. Then the derivatives with respect to the variable ξ become the derivatives with respect to the variable φ as

$$\frac{\mathrm{d}}{\mathrm{d}\xi} = \varepsilon \sqrt{\sum_{j=0}^{r} c_j \varphi^j}\, \frac{\mathrm{d}}{\mathrm{d}\varphi} \tag{2.6}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} = \frac{1}{2} \sum_{j=1}^{r} j c_j \varphi^{j-1} \frac{\mathrm{d}}{\mathrm{d}\varphi} + \sum_{j=0}^{r} c_j \varphi^j \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}, \cdots. \tag{2.7}$$

Step 3. Determine the parameters n, r and c, a_i, c_j $(i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, r)$. Substituting (2.4) into (2.2) and balancing the highest derivative term with the nonlinear terms by making use of (2.6) and (2.7), we then obtain a relation for n and r, from which the different possible values of n and r can be obtained. These values lead to the different series expansions of the exact solutions for equation (2.1). For example, in the case of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

we have

$$r = n + 2. \tag{2.8}$$

If we take $n = 1$ and $r = 3$ in (2.8), we may use the following series expansion as a solution of the KdV equation

$$u = a_0 + a_1\varphi \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3}.$$

Similarly, if we take $n = 2$, $r = 4$ in (2.8), we have

$$u = a_0 + a_1\varphi + a_2\varphi^2 \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}.$$

Substituting the expansion (2.4) into equation (2.2) and putting the same powers of φ^i and $\varphi^i\sqrt{\sum_{j=0}^{r} c_j\varphi^j}$ together, we obtain a polynomial about φ. Because of the independence of functions φ^i $(i = 0, 1, \ldots)$, we may set their coefficients to zero and get a system of algebraic equations, from which the above parameters can be found explicitly.

Step 4. Solve for equation (2.5). Substituting the parameters c_j $(j = 0, 1, \ldots, r)$ obtained in step 3 into equation (2.5), we can then obtain all the possible solutions. We remark here that the solutions of equation (2.1) depend on the explicit solvability of equation (2.5). The solution of the system of algebraic equations will be getting tedious with the increase of the values of n and r. In this case when $r = 4$, equation (2.5) gives a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. We consider only the case $r = 4$ in this paper and have

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}. \tag{2.9}$$

By considering the different values of c_0, c_1, c_2, c_3 and c_4, we find that equation (2.9) admits a series of fundamental solutions, which are classified as follows.

Case A. Equation (2.9) admits two kinds of polynomial solutions as follows:

$$\varphi = \varepsilon\sqrt{c_0}\xi \qquad \text{as} \quad c_1 = c_2 = c_3 = c_4 = 0 \quad c_0 > 0 \tag{2.10}$$

and

$$\varphi = -\frac{c_0}{c_1} + \frac{1}{4}c_1\xi^2 \qquad \text{as} \quad c_2 = c_3 = c_4 = 0 \quad c_1 \neq 0. \tag{2.11}$$

Case B. Equation (2.9) possesses two kinds of exponential solutions, namely,

$$\varphi = -\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}\xi) \qquad \text{as} \quad c_3 = c_4 = 0 \quad c_0 = \frac{c_1^2}{4c_2} \quad c_2 > 0 \tag{2.12}$$

and

$$\varphi = \frac{c_3}{2c_4}\exp\left(\frac{\varepsilon c_3}{2\sqrt{-c_4}}\xi\right) \qquad \text{as} \quad c_0 = c_1 = c_2 = 0 \quad c_4 < 0. \tag{2.13}$$

Case C. Equation (2.9) admits six kinds of triangular solutions as follows:

$$\varphi = -\frac{c_1}{2c_2} + \frac{\varepsilon c_1}{2c_2}\sin(\sqrt{-c_2}\xi) \qquad \text{as} \quad c_0 = c_3 = c_4 = 0 \quad c_2 < 0 \tag{2.14}$$

$$\varphi = \varepsilon\sqrt{-\frac{c_0}{c_2}}\sin(\sqrt{-c_2}\xi) \qquad \text{as} \quad c_1 = c_3 = c_4 = 0 \quad c_0 > 0 \quad c_2 < 0 \tag{2.15}$$

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\sec(\sqrt{-c_2}\xi) \qquad \text{as} \quad c_0 = c_1 = c_3 = 0 \quad c_2 < 0 \quad c_4 > 0 \tag{2.16}$$

$$\varphi = -\frac{c_2}{c_3}\sec^2\left(\frac{\sqrt{-c_2}}{2}\xi\right) \qquad \text{as} \quad c_0 = c_1 = c_4 = 0 \quad c_2 < 0 \tag{2.17}$$

$$\varphi = \varepsilon\sqrt{\frac{c_2}{2c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right) \qquad \text{as} \quad c_1 = c_3 = 0 \quad c_0 = \frac{c_2^2}{4c_4} \quad c_2 > 0 \quad c_4 > 0 \tag{2.18}$$

and

$$\varphi = -\frac{c_2\sec^2\left(\frac{1}{2}\sqrt{-c_2}\xi\right)}{2\varepsilon\sqrt{-c_2c_4}\tan\left(\frac{1}{2}\sqrt{-c_2}\xi\right) + c_3} \qquad \text{as} \quad c_0 = c_1 = 0 \quad c_2 < 0. \tag{2.19}$$

In the case when $c_4 = 0$, the solution (2.19) degenerates to the solution (2.17).

Case D. Equation (2.9) admits six kinds of hyperbolic solutions, namely,

$$\varphi = -\frac{c_1}{2c_2} + \frac{\varepsilon c_1}{2c_2}\sinh(2\sqrt{c_2}\xi) \qquad \text{as} \quad c_0 = c_3 = c_4 = 0 \quad c_2 > 0 \tag{2.20}$$

$$\varphi = \varepsilon\sqrt{\frac{c_0}{c_2}}\sinh(\sqrt{c_2}\xi) \qquad \text{as} \quad c_1 = c_3 = c_4 = 0 \quad c_0 > 0 \quad c_2 > 0 \tag{2.21}$$

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}(\sqrt{c_2}\xi) \qquad \text{as} \quad c_0 = c_1 = c_3 = 0 \quad c_2 > 0 \quad c_4 < 0 \tag{2.22}$$

$$\varphi = -\frac{c_2}{c_3}\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right) \qquad \text{as} \quad c_0 = c_1 = c_4 = 0 \quad c_2 > 0 \tag{2.23}$$

$$\varphi = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad \text{as} \quad c_1 = c_3 = 0 \quad c_0 = \frac{c_2^2}{4c_4} \quad c_2 < 0 \quad c_4 > 0 \tag{2.24}$$

and

$$\varphi = \frac{c_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\varepsilon\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} \qquad \text{as} \quad c_0 = c_1 = 0 \quad c_2 > 0. \tag{2.25}$$

In the case when $c_4 = 0$, the solution (2.25) degenerates to the solution (2.23). As $c_3 = 2\varepsilon\sqrt{c_2c_4}$, the solution (2.25) degenerates to the following solution:

$$u = \frac{1}{2}\varepsilon\sqrt{\frac{c_2}{c_4}}\left[1 + \tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)\right]$$

which is the same kind of solution with (2.24).

Case E. Equation (2.9) admits two kinds of rational solutions, namely,

$$\varphi = -\frac{\varepsilon}{\sqrt{c_4}\xi} \qquad \text{as} \quad c_0 = c_1 = c_2 = c_3 = 0 \quad c_4 > 0 \tag{2.26}$$

and

$$\varphi = \frac{4c_3}{c_3^2\xi^2 - 4c_4} \qquad \text{as} \qquad c_0 = c_1 = c_2 = 0. \tag{2.27}$$

Case F. Equation (2.9) admits three Jacobi elliptic doubly periodic wave solutions as follows:

$$\varphi = \sqrt{-\frac{c_2 m^2}{c_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_1 = c_3 = 0$$
$$c_0 = \frac{c_2^2 m^2(m^2-1)}{c_4(2m^2-1)^2} \qquad c_2 > 0 \qquad c_4 < 0 \tag{2.28}$$

$$\varphi = \sqrt{\frac{-c_2}{c_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_1 = c_3 = 0$$
$$c_0 = \frac{c_2^2(1-m^2)}{c_4(m^2-2)^2} \qquad c_2 > 0 \qquad c_4 < 0 \tag{2.29}$$

and

$$\varphi = \varepsilon\sqrt{-\frac{c_2 m^2}{c_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad \text{as} \qquad c_1 = c_3 = 0$$
$$c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2} \qquad c_2 < 0 \qquad c_4 > 0 \tag{2.30}$$

where m is a modulus. The Jacobi elliptic functions are doubly periodical and possess properties of triangular functions, namely,

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1 \qquad \mathrm{dn}^2\xi = 1 - m^2\mathrm{sn}^2\xi$$
$$(\mathrm{sn}\,\xi)' = \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi \qquad (\mathrm{cn}\,\xi)' = -\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi \qquad (\mathrm{dn}\,\xi)' = -m^2\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\,\xi \to \tanh\xi \qquad \mathrm{cn}\,\xi \to \mathrm{sech}\,\xi \qquad \mathrm{dn}\,\xi \to \mathrm{sech}\,\xi.$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}\,\xi \to \sin\xi \qquad \mathrm{cn}\,\xi \to \cos\xi \qquad \mathrm{dn}\,\xi \to 1.$$

More detailed notation for the Weierstrass and Jacobi elliptic functions can be found in [27, 28].

Let us simply show formulae (2.28)–(2.30). In the case when $c_3 = c_1 = 0$, by using the transformations

$$c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2} \qquad \bar{\varphi} = \sqrt{-\frac{c_4(m^2+1)}{c_2 m^2}}\varphi \qquad \bar{\xi} = \sqrt{-\frac{c_2}{m^2+1}}\xi$$

equation (2.9) is reduced to the equation

$$\bar{\varphi}' = \pm\sqrt{1-(m^2+1)\bar{\varphi}^2 + m^2\bar{\varphi}^4}$$

which has a Jacobi elliptic solution $\bar{\varphi} = \mathrm{sn}(\xi, m)$. Therefore, we can obtain the solution (2.30). Again formulae (2.28) and (2.29) can be obtained in a similar way. As $m \to 1$, the Jacobi doubly periodic solutions (2.28) and (2.29) degenerate to the solitary wave solutions (2.20), and (2.30) degenerates to the solitary wave solutions (2.24).

Case G. Equation (2.9) admits a Weierstrass elliptic doubly periodic-type solution

$$\varphi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) \qquad \text{as} \quad c_2 = c_4 = 0 \quad c_3 > 0 \tag{2.31}$$

where $g_2 = -4c_1/c_3$, and $g_3 = -4c_0/c_3$ are called invariants of the Weierstrass elliptic function. In fact, when $c_2 = c_4 = 0$ in equation (2.9) by using transformations

$$\bar{\xi} = \frac{\sqrt{c_3}}{2}\xi \qquad c_0 = -\frac{1}{4}c_3 g_3 \qquad c_1 = -\frac{1}{4}c_3 g_2$$

equation (2.9) becomes

$$\varphi'_{\bar{\xi}} = \pm\sqrt{-g_3 - g_2\varphi + 4\varphi^3}$$

which has a Weierstrass elliptic doubly periodic solution $\varphi = \wp(\bar{\xi}, g_2, g_3)$.

Remark 1. The other types of travelling wave solutions such as cosec ξ, cot ξ, cosech ξ and coth ξ can also be obtained from equation (2.9). These solutions appear in pairs of the functions sec ξ, tan ξ, sech ξ and tanh ξ, respectively. Since the solutions (2.16)–(2.19), (2.26) and (2.27) make the solutions of an equation under investigation diverge, we omit them in this paper.

Remark 2. Let us consider three special cases of our proposed method. In the case $c_1 = c_3 = 0, c_0 = 1, c_2 = -2, c_4 = 1$, equation (2.9) has a solution tanh ξ and our method reduces to the tanh method [16–18]. In the case when $c_1 = c_3 = 0, c_0 = c_2^2/4, c_4 = 1$, equation (2.9) degenerates to a Riccati equation. In this case our proposed method becomes the extended tanh method [20, 21]. The cases (2.28)–(2.30) readily cover the results of the Jacobi function expansion method [29, 30]. In conclusion, our proposed method is a generalization of either the tanh method or the extended tanh method.

The proposed method not only gives a unified formulation to uniformly construct a series of exact solutions, but also provides a guideline to classify the types of solutions according to the given parameters. Furthermore, the proposed method is computerizable in solving nonlinear equations by using symbolic software like *Mathematica* or *Maple*.

3. Application

In this section, we apply the method developed in section 2 to various nonlinear equations and give their series of exact solutions.

3.1. Integrable and nonintegrable equations

Example 1. Recently by considering the extension of (1 + 1)-dimensional Calogero–KdV equation [31, 32], Yu and Toda introduced a new (2 + 1)-dimensional Calogero–KdV equation

$$u_t + \frac{1}{4}u_{xxy} + \frac{u_y}{4u^2} + \frac{1}{8}u_x\partial_x^{-1}\left(\frac{1}{u^2}\right)_y + \frac{u_x^2u_y}{2w^2} - \frac{1}{8}u_x\partial_x^{-1}\left(\frac{u_x^2}{u^2}\right)_y - \frac{u_xu_{xy}}{2u} - \frac{u_{xx}u_y}{4u} = 0. \tag{3.1}$$

They further showed that equation (3.1) was integrable in the sense of the Painlevé property [33]. Our proposed method in this paper will give a series of new explicit exact solutions to equation (3.1) as follows.

By considering the wave transformations $u = U(\xi), \xi = x + cy + dt$, we change the equation (3.1) to the form

$$8dU^2U' + 2cU^2U''' + 3cU' + 3cU'^3 - 6cUU'U'' = 0. \tag{3.2}$$

According to the proposed method, we use the following series expansion as solutions of equation (3.2):

$$U = \sum_{i=0}^{n} a_i\varphi^i(x, y, t) = \sum_{i=0}^{n} a_i\varphi^i(\xi)$$

where $\varphi(\xi)$ satisfies equation (2.5). Balancing the term U^2U''' with term $UU'U''$ in (3.2) gives

$$3n-3+r=3n-1-2+r$$

which implies that n and r are arbitrary. We take $r=4$ and $n=2$ and have

$$U=a_0+a_1\varphi+a_2\varphi^2 \tag{3.3}$$

where φ satisfies (2.9).

Substituting (3.3) into (3.2) and using *Mathematica* gives the following system:

$$3\varepsilon ca_1+8\varepsilon da_0^2a_1+3\varepsilon^3ca_1^3c_0-12\varepsilon^3ca_0a_1a_2c_0-3\varepsilon^3ca_0a_1^2c_1+6\varepsilon^3ca_0^2a_2c_1+2\varepsilon^3ca_0^2a_1c_2=0$$
$$8\varepsilon da_0a_1^2+3\varepsilon ca_2+8\varepsilon da_0^2a_2+3\varepsilon^3ca_1^2a_2c_0-12\varepsilon^3ca_0a_2^2c_0-6\varepsilon^3ca_0a_1a_2c_1-\varepsilon^3ca_0a_1^2c_2$$
$$+8\varepsilon^3ca_0^2a_2c_2+3\varepsilon^3ca_0^2a_1c_3=0$$
$$8\varepsilon da_1^3+48\varepsilon da_0a_1a_2-3\varepsilon^3ca_1^2a_2c_1-24\varepsilon^3ca_0a_2^2c_1-\varepsilon^3ca_1^3c_2+3\varepsilon^3ca_0a_1^2c_3$$
$$+30\varepsilon^3ca_0^2a_2c_3+12\varepsilon^3ca_0^2a_1c_4=0$$
$$8\varepsilon da_1^2a_2+8\varepsilon da_0a_2^2-3\varepsilon^3ca_1a_2^2c_1-\varepsilon^3ca_1^2a_2c_2-4\varepsilon^3ca_0a_2^2c_2+6\varepsilon ca_0a_1a_2c_3$$
$$+3\varepsilon^3ca_0a_1^2c_4+12\varepsilon^3ca_0^2a_2c_4=0$$
$$40\varepsilon da_1a_2^2-6\varepsilon^3ca_2^3c_1-14\varepsilon^3ca_1a_2^2c_2+3\varepsilon^3ca_1^2a_2c_3+3\varepsilon^3cc_4+60\varepsilon^3ca_0a_1a_2c_4=0$$
$$8\varepsilon da_2^3-4\varepsilon^3cc_2-3\varepsilon^3ca_1a_2^2c_3+9\varepsilon^3ca_1^2a_2c_4+12\varepsilon^3ca_0a_2^2c_4=0$$
$$-\varepsilon^3ca_2^3c_3+2\varepsilon^3ca_1a_2^2c_4=0.$$

Note that $\varepsilon=\pm1$ and hence $\varepsilon^3=\varepsilon$. We may eliminate ε from the above system. From the output of *Mathematica*, we find three kinds of solutions, namely,

$$c_1=c_3=a_1=0 \qquad c_4=\frac{a_2(cc_2-2d)}{3ca_0} \qquad a_2=\frac{3c+8da_0^2+8ca_0^2c_2}{12ca_0c_0} \tag{3.4}$$

with $a_0, c_0, c\neq0, c_2$ and d being arbitrary constants,

$$c_4=a_2=0 \qquad c_3=\frac{a_1(cc_2-8d)}{3ca_0} \qquad c_1=\frac{3c+8da_0^2+3ca_1^2c_0+2ca_0^2c_2}{3ca_0a_1} \tag{3.5}$$

with $a_0, a_1, c\neq0, c_0, c_2$ and d being arbitrary constants and

$$\begin{gathered} c_0=c_1=0 \qquad c_2=-\frac{a_1^2}{a_0^2\left(a_1^2-4a_0a_2\right)} \qquad c_3=-\frac{2a_1a_2}{a_0^2\left(a_1^2-4a_0a_2\right)} \\ c_4=-\frac{a_2^2}{a_0^2\left(a_1^2-4a_0a_2\right)} \qquad d=-\frac{c\left(a_1^2-12a_0a_2\right)}{8a_0^2\left(a_1^2-4a_0a_2\right)} \end{gathered} \tag{3.6}$$

with a_0, a_1, a_2 and c being arbitrary constants.

Since $c_1=c_3=0$ in (3.4), by using (2.22), (2.28), (2.29) and (3.4), we obtain two kinds of solutions, namely, a line solitary wave solution

$$u_1=a_0-\frac{3ca_0c_2}{cc_2-2d}\operatorname{sech}^2(\sqrt{c_2}\xi) \qquad c_2>0$$

and a Jacobi doubly periodic solution

$$u_2=a_0-\frac{3m^2ca_0c_2}{(2m^2-1)(cc_2-2d)}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)$$

where $\xi=x+cy+dt$. As $m\to1$, the Jacobi doubly periodic solution u_2 degenerates to the line solitary wave solution u_1.

From (2.21), (2.29) and (3.4), and using a simple transformation $c_2 \to -2c_2$, we find that the corresponding solutions are the same with u_1 and u_2, respectively.

Setting $c_4 = 0$ in (3.5), then by (2.9) and (2.15), the obtained solutions are the same with u_1, u_2 by using the transformations $c_2 \to 4c_2$ and $a_1 \to a_2$.

From (2.31) and (3.6), we get a Weierstrass periodic solution

$$u_3 = a_0 + a_1\wp\left(\sqrt{\frac{-2a_1d}{3ca_0}}\xi, g_2, g_3\right)$$

where $\xi = x + cy + dt$ and

$$g_2 = \frac{3c + 3ca_1^2c_0 + 8da_0^2}{2a_1^2d} \qquad g_3 = \frac{3ca_0c_0}{2a_1c}.$$

We take $d = cc_2/8$, then (3.5) becomes

$$c_3 = c_4 = a_2 = 0 \qquad c_1 = \frac{1 + a_1^2c_0 + a_0^2c_2}{a_0a_1}. \tag{3.7}$$

If we restrict $c_0 = c_1^2/(4c_2)$ in (3.7), by using (2.12) we obtain an exponential-type solution as follows:

$$u_4 = a_0 - \frac{c_1}{2c_2} + a_1\exp(\pm\sqrt{c_2}\xi)$$

where $\xi = x + cy + cc_2t/8$ and $4c_2\big(a_0a_1c_1 - a_0^2c_2 - 1\big) - a_1^2c_1^2 = 0$.

If we restrict $c_0 = 0$ in (3.7), by using (2.20) and (2.21) we obtain a triangular-type solution

$$u_5 = a_0 - \frac{1 + a_0^2c_2}{2a_0a_1c_2}[1 \pm \sin(\sqrt{-c_2}\xi)] \qquad c_2 < 0$$

and a hyperbolic-type solution

$$u_6 = a_0 - \frac{1 + a_0^2c_2}{2a_0a_1c_2}[1 \pm \sinh(2\sqrt{c_2}\xi)] \qquad c_2 > 0$$

where $\xi = x + cy + cc_2t/8$.

From (2.19) and (3.6), we find that the obtained solitary wave solution is the same kind with u_1. In addition, the solutions u_4 and u_6 are nonlocalized solutions. We remark here that the solutions u_1 and u_2 can be obtained by applying the tanh method [16, 17] and Jacobi function expansion method [29, 30], respectively. But the solutions u_4, u_5 and u_6 cannot be obtained by these methods.

Example 2. Consider the $(3+1)$-dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3(uu_y)_x + 3u_{xx} + 3u_{xx}\partial_x^{-1}u_y + 3u_xu_y + 2u_{yt} - 3u_{xz} = 0$$

which describes certain physically interesting $(3+1)$-dimensional waves but does not pass any of the conventional integrability tests [34, 35]. A kind of solitary wave solution was further given by Hong and Oh recently [36]. Here, similar to example 1, our proposed method gives a series of exact solutions as follows:

a line solitary wave solution

$$u_1 = a_0 + 2c_2\operatorname{sech}^2(\sqrt{c_2}\xi) \qquad c_2 > 0$$

a Jacobi doubly periodic solution

$$u_2 = a_0 + \frac{2m^2c_2}{2m^2 - 1}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2 - 1}}\xi\right) \qquad c_2 > 0$$

and a Weierstrass periodic solution

$$u_3 = a_0 + a_1 \wp\left(\sqrt{\frac{-a_1}{2}}\xi, g_2, g_3\right) \qquad \text{with} \qquad c_2 = 0$$

where $\xi = x + cy + \frac{2c}{3}(e + 3a_0 + 2c_2) + et$ and $g_2 = 2c_1/a_1,\ g_3 = 2c_0/a_1$. As $m \to 1$, the Jacobi periodic solution u_2 degenerates to the line solitary wave solution u_1.

Example 3. Consider the symmetric regularized long wave equation

$$u_{tt} + u_{xx} + uu_{xt} + u_x u_t + u_{xxtt} = 0$$

which arises in several physical applications including ion sound waves in a plasma [37]. The Painlevé test predicts that the symmetric regularized long wave equation is not solvable by inverse scattering [38]. In a similar way to examples 1 and 2, we obtain three kinds of solutions, namely, a solitary wave solution

$$u_1 = -\frac{1}{c}(1 + c^2 + 4c^2c_2) + 12cc_2\operatorname{sech}^2(\sqrt{c_2}\xi_1) \qquad c_2 > 0$$

a Jacobi doubly periodic solution

$$u_2 = -\frac{1}{c}(1 + c^2 + 4c^2c_2) + \frac{12cc_2m^2}{c_4(2m^2 - 1)}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2 - 1}}\xi_1\right)$$

and a Weierstrass doubly periodic solution

$$u_3 = -\frac{1}{c}(1 + c^2 + 4c^2c_2) + a_1\wp\left(\sqrt{\frac{-a_1}{12c}}\xi_3, g_2, g_3\right)$$

where

$$\xi = x + ct \qquad g_2 = \frac{12cc_1}{a_1} \qquad g_3 = \frac{12cc_0}{a_1}.$$

As $m \to 1$, the Jacobi periodic solution u_2 degenerates to the line solitary wave solution u_1.

3.2. Systems of nonlinear evolution equations

In general, the system of nonlinear equations is more difficult to solve than a single equation, especially by using direct integration methods. Our proposed method works equally well for such systems without much extra effort compared with single equations.

Example 4. Consider the coupled Drinfel'd–Sokolov–Wilson equation

$$\begin{aligned} &u_t + 3vv_x = 0 \\ &v_t + 2v_{xxx} + 2uv_x + u_x v = 0 \end{aligned} \tag{3.8}$$

which was introduced as a model of water waves [39, 40]. Solitary wave solution and solitary wave structure of this system were investigated [22, 41]. Here our proposed method gives a series of travelling wave solutions to the system as follows.

Using transformation $u = U(\xi)$, $v = V(\xi)$, $\xi = x + ct$, we reduce equation (3.8) to the following system:

$$\begin{aligned} &cU' + 3VV' = 0 \\ &cV' + 2V''' + 2UV' + U'V = 0. \end{aligned} \tag{3.9}$$

We expand the solutions of equation (3.9) as

$$H = \sum_{i=0}^{n_1} a_i \varphi^i(\xi) \qquad U = \sum_{j=0}^{n_2} b_j \varphi^j(\xi)$$

where φ satisfies (2.5). Balancing the highest derivative terms with nonlinear terms in (3.9) gives

$$n_1 = 2n_2 \qquad r = n_2 + 2.$$

Therefore, we may choose $n_1 = 2, n_2 = 1, r = 4$ and have

$$U = a_0 + a_1\varphi + a_2\varphi^2 \qquad U = b_0 + b_1\varphi \tag{3.10}$$

where φ satisfies (2.9).

Substituting (3.10) into (3.9) and using *Mathematica*, we obtain a system of algebraic equations:

$$\begin{aligned}
&\varepsilon c a_1 + 3\varepsilon b_0 b_1 = 0\\
&2\varepsilon c a_2 + 3\varepsilon b_1^2 = 0\\
&\varepsilon a_1 b_0 + \varepsilon c b_1 + 2\varepsilon a_0 b_1 + 2\varepsilon^3 b_1 c_2 = 0\\
&2\varepsilon a_2 b_0 + 3\varepsilon a_1 b_1 + 6\varepsilon^3 b_1 c_3 = 0\\
&4\varepsilon a_2 b_1 + 12\varepsilon^3 b_1 c_4 = 0.
\end{aligned}$$

Note that $\varepsilon = \pm 1$ and hence $\varepsilon^3 = \varepsilon$. We may eliminate ε from the above system. Solving the system, we obtain two kinds of solutions, namely,

$$c_1 = c_3 = a_1 = b_0 = 0 \qquad a_0 = -\frac{1}{2}(c + 2c_2) \qquad a_2 = -\frac{3b_1}{2c} \qquad c_4 = \frac{b_1^2}{2c} \tag{3.11}$$

with b_1, c_0, c_2 and c being arbitrary constants, and

$$\begin{aligned}
&c_0 = c_1 = 0 \qquad a_0 = -\frac{1}{2c}(c^2 + 2cc_2 - 3b_0^2) \qquad a_1 = -\frac{3b_0 b_1}{c}\\
&a_2 = -\frac{3b_1^2}{2c} \qquad c_3 = \frac{2b_0 b_1}{c} \qquad c_4 = \frac{b_1^2}{2c}
\end{aligned} \tag{3.12}$$

where b_0, b_1, c_2 and c are arbitrary constants.

From (2.22), (2.24) and (3.11), we obtain two kinds of solitary wave solutions, namely,

$$\begin{aligned}
&u_1 = -\tfrac{1}{2}(c + 2c_2) + 3c_2 \operatorname{sech}^2(\sqrt{c_2}\xi)\\
&v_1 = \sqrt{-cc_2}\operatorname{sech}(\sqrt{c_2}\xi) \qquad c_2 > 0
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&u_2 = -\frac{1}{2}(c + 2c_2) + \frac{3}{2}c_2 \tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right)\\
&v_2 = \pm\sqrt{-cc_2}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0
\end{aligned} \tag{3.14}$$

where $\xi = x + ct$.

Again from (2.28)–(2.30) and (3.11), we also obtain three Jacobi doubly periodic solutions, namely,

$$\begin{aligned}
&u_3 = -\frac{1}{2}(c + 2c_2) + \frac{3c_2 m^2}{2m^2 - 1}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2 - 1}}\xi\right)\\
&v_3 = \sqrt{\frac{-cc_2 m^2}{2m^2 - 1}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2 - 1}}\xi\right) \qquad c_2 < 0
\end{aligned} \tag{3.15}$$

$$u_4 = -\frac{1}{2}(c+2c_2) + \frac{3c_2}{2-m^2}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)$$
$$v_4 = \sqrt{\frac{-cc_2}{2-m^2}}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2 < 0 \tag{3.16}$$

and

$$u_5 = -\frac{1}{2}(c+2c_2) + \frac{3c_2m^2}{m^2+1}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$
$$v_5 = \pm\sqrt{\frac{-2cc_2m^2}{m^2+1}}\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2 < 0 \tag{3.17}$$

where $\xi = x+ct$. As $m \to 1$, the Jacobi doubly periodic solutions (3.15) and (3.16) degenerate to the solitary wave solutions (3.13), and (3.17) degenerates to (3.14).

Setting $c_2 = 0$ in (3.12) and using (2.13), we then obtain an exponential solution

$$u_6 = -\frac{1}{2c}\left(c^2 - 3b_0^2\right) - \frac{6b_0^2}{c}\varphi - \frac{3b_0b_1}{c}\varphi^2$$
$$v_6 = b_0 + 2b_0\varphi \qquad \varphi = \exp\left(\pm b_0\sqrt{-\frac{2}{c}}\xi\right) \qquad c < 0. \tag{3.18}$$

From (2.25) and (3.12), we obtain a solitary wave solution

$$u_7 = -\frac{1}{2c}\left(c^2 + 2cc_2 - 3b_0^2\right) - \frac{3b_0b_1}{c}\varphi - \frac{3b_1^2}{2c}\varphi^2$$
$$v_7 = b_0 + b_1\varphi \qquad \varphi = \frac{2cc_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\pm b_1\sqrt{2cc_2}\tanh\xi - 2b_0b_1} \tag{3.19}$$

where $\xi = x + ct$. Here we remark that the solution (3.14) can be obtained by the tanh method and the extended method [16, 20], and (3.13) and (3.14) were obtained by the mixed exponential method [22]. The other five solutions (3.15)–(3.19) are new and cannot be obtained by these methods.

Example 5. Consider the $(2+1)$-dimensional generalized dispersive long wave equation

$$u_{ty} + (v_x + uu_y)_x = 0$$
$$v_t + (uv + u + u_{xy})_x = 0 \tag{3.20}$$

which was introduced by Ablowitz and Clarkson [1]. In the $(1+1)$-dimensional reduce $x = y$, the system (3.20) becomes the dispersive long wave equation, which is known to be completely integrable [42, 43]. The proposed method gives a series of new travelling wave solutions for the system.

In a way similar to example 5, we obtain two solitary wave solutions

$$u_1 = -c \pm \sqrt{-2c_2}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)$$

$$v_1 = -(1+cc_2) + cc_2\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0$$

and

$$u_2 = -\frac{b_1 + a_1 cd}{ca_1} + a_1\varphi$$
$$v_2 = \frac{b_1^2 - ca_1^2 - c^2a_1^2c_2}{ca_1^2} + b_1\varphi - \frac{1}{2}ca_1^2\varphi^2 \qquad c_2 > 0$$

where

$$\varphi = \frac{c_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\pm a_1\sqrt{c_2}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) + b_1} \qquad \xi = x + cy + dt.$$

A Jacobi doubly periodic solution reads

$$u_3 = -c \pm 2\sqrt{-\frac{c_2m^2}{m^2+1}}\operatorname{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$
$$v_3 = -c_2 + \frac{2c_2m^2}{m^2+1}\operatorname{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2 < 0$$

where $\xi = x + cy + dt$. As $m \to 1$, the Jacobi periodic solutions (u_3, v_3) degenerate to the solitary wave solution (u_1, v_1). We remark that the solitary wave solution can be obtained by the tanh method and mixed exponential method.

3.3. Special-type equations

It is well known that physics and engineering often provide special types of nonlinear equations such as the sine-Gordon equation, sinh-Gordon equation and Schrödinger equation. In the following, our proposed method is extended to such equations whose solutions require some kinds of 'prepossessing' techniques.

Example 6. We consider the double sine-Gordon equation

$$u_{xt} = \sin u + \sin(2u) \tag{3.21}$$

which is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves and smectic liquid crystal dynamics [44–48]. Though the sine-Gordon equation

$$u_{xt} = \sin u$$

which is completely integrable, the Painlevé test and numerical evidence predict that doubly sine-Gordon equation (3.21) is not thought to be completely integrable [1]. The known travelling wave solution to equation (3.21) is

$$u = 2\arctan\left[\sqrt{3}\tanh\left(cx - \frac{3}{8c}t\right)\right] \tag{3.22}$$

and it is the aim of our proposed method to find a more general solution, including (3.22).

To extend our proposed method to equation (3.21), we consider the transformations

$$u = 2\arctan v \qquad v = V(\xi) \qquad \xi = x + ct$$

and hence have

$$u_{xt} = \frac{2c(V^2V'' + V'' - 2VV'^2)}{(1+V^2)^2} \qquad \sin u = \frac{2V}{1+V^2} \qquad \sin(2u) = \frac{4V(1-V^2)}{(1+V^2)^2}. \tag{3.23}$$

Substituting (3.23) into (3.21), equation (3.21) is reduced to a polynomial-type equation

$$c(1+V^2)V'' - 2cVV'^2 - 3V + V^3 = 0. \tag{3.24}$$

We expand the solution of equation (3.24) in the form

$$U = \sum_{i=0}^{n} a_i \varphi^i(x,t) = \sum_{i=0}^{n} a_i \varphi^i(\xi)$$

where φ satisfies equation (2.5). Balancing the term V^2V'' with the term VV'^2 in (3.24) gives

$$n + 2n - 2 + r = 2n + n - 2 + r$$

from which we see that n and r are arbitrary. We take $r = 4$ and $n = 1$ and have

$$U = a_0 + a_1\varphi \tag{3.25}$$

where φ satisfies (2.9).

Substituting (3.25) into (3.24) and using *Mathematica* yields a system of algebraic equations:

$$\begin{aligned}
&-3a_1 + 3a_0^2a_1 - 2\varepsilon^2 c a_1^3 c_0 - \varepsilon^2 c a_0 a_1^2 c_1 + \varepsilon^2 c a_1 c_2 + \varepsilon^2 c a_0^2 a_1 c_2 = 0\\
&a_1^3 - \varepsilon^2 c a_1^2 c_2 + \varepsilon^2 c a_0 a_1^2 c_3 + 2\varepsilon^2 c a_1 c_4 + 2\varepsilon^2 c a_0^2 a_1 c_4 = 0\\
&-6a_0 + 2a_0^3 - 4\varepsilon^2 c a_0 a_1^2 c_0 + \varepsilon^2 c a_1 c_1 + \varepsilon^2 c a_0^2 a_1 c_1 = 0\\
&2a_0 a_1^2 - \varepsilon^2 c a_1^3 c_1 + \varepsilon^2 c a_1 c_3 + \varepsilon^2 c a_0^2 a_1 c_3 = 0\\
&-\varepsilon^2 c a_1^3 c_3 + 4\varepsilon^2 c a_0 a_1^2 c_4 = 0.
\end{aligned}$$

After eliminating ε, we solve the above system and obtain three kinds of solutions, namely,

$$c_3 = c_1 = a_0 = 0 \qquad c = -\frac{3}{2a_1^2c_0 - c_2} \qquad c_4 = \frac{1}{3}a_1^2\left(a_1^2c_0 + c_2\right) \tag{3.26}$$

with a_1, c_0 and c_2 being arbitrary constants,

$$c_0 = c_3 = c_4 = 0 \qquad a_0 = \frac{a_0^2 - 1}{a_1^2 c} \qquad c_1 = \frac{2a_0}{a_1 c} \qquad c_2 = \frac{1}{c} \tag{3.27}$$

with a_1 and c being arbitrary constants and

$$c_0 = c_1 = 0 \qquad a_0 = \pm\sqrt{3} \qquad c_2 = -\frac{3}{2c} \qquad c_3 = \mp\frac{\sqrt{3}a_1}{2c} \qquad c_4 = -\frac{a_1^2}{8c} \tag{3.28}$$

where a_1 and c are arbitrary constants.

Taking $c_0 = c_2^2/4c_4$ in (3.26) gives

$$c = \frac{3}{4c_2} \qquad c_4 = -\frac{1}{6}a_1^2 c_2.$$

In this case, by using (2.20) and (2.30), we obtain a solitary wave solution

$$u_1 = 2\arctan\left\{\pm\sqrt{3}\tanh(\sqrt{-\frac{c_2}{2}}(x + \frac{3}{4c_2}t)\right\} \qquad c_2 < 0$$

and a Jacobi doubly periodic solution

$$u_2 = 2\arctan\left\{\pm\sqrt{\frac{6m^2}{m^2+1}}\mathrm{sn}(\sqrt{-\frac{c_2}{m^2+1}}(x + \frac{3}{4c_2}t)\right\} \qquad c_2 < 0.$$

From (2.14), (2.20) and (3.27), we obtain a triangular periodic solution

$$u_3 = 2\arctan\{\pm\sin[\sqrt{-1/c}(x + ct)]\} \qquad c < 0$$

and a hyperbolic solution

$$u_4 = 2\arctan\{\pm\sinh[2\sqrt{1/c}(x+ct)]\} \qquad c > 0.$$

From (2.25) and (3.28), we find that the corresponding solution is the same as the solution u_1 since $c_3^2 = 4c_2c_4$. We remark that the solutions u_3 and u_4 cannot be obtained by the tanh method and the Jacobi function expansion method.

Example 7. Consider the Calogero–Degasperis–Fokas equation [32, 49]

$$u_t = u_{xxx} - \tfrac{1}{8}u_x^3 + (\alpha\mathrm{e}^u + \beta\mathrm{e}^{-u} + \gamma)u_x \tag{3.29}$$

which is completely integrable and solvable by the inverse scattering [1].

We make transformation $u = \ln v$, $v = V(\xi) = V(x+ct)$, then equation (3.29) becomes

$$-8\alpha V^3V' - 17V'^3 - 8\beta VV' + 24VV'V'' - 8(\gamma - c)V^2V' - 8V^2V''' = 0. \tag{3.30}$$

Substituting the expansion

$$V = a_0 + a_1\varphi + a_2\varphi^2$$

into equation (3.30), we may obtain a system of algebraic equations, from whose solutions and using (2.10)–(2.31), we obtain a solitary wave solution

$$u_1 = \ln\left\{\sqrt{\frac{\beta}{\alpha}}\tanh^2\left(\frac{4\alpha\beta}{25}\right)^{1/4}\left[x + \left(\gamma + \frac{2}{5}\sqrt{\alpha\beta}\right)t\right]\right\}$$

a Jacobi doubly periodic wave solution

$$u_2 = \ln\left\{\sqrt{\frac{2\beta m^2}{\alpha(m^2+1)}}\mathrm{sn}^2\left(\frac{8\alpha\beta}{25(m^2+1)}\right)^{1/4}\left[x + \left(\gamma + \frac{2}{5}\sqrt{\alpha\beta}\right)t\right]\right\}$$

an exponential solution

$$u_3 = \ln\left\{a_1\exp\left(\pm\sqrt{\frac{8\beta}{5a_0}}\left[x + \left(\gamma + \frac{\beta}{5a_0}\right)t\right]\right)\right\} \qquad \alpha = 0$$

and Weierstrass doubly periodic solution

$$u_4 = \ln\left\{a_0 + a_1\wp\left(\sqrt{-\frac{2\alpha a_1}{5}}\left[x + \left(\gamma + \frac{3}{5}\alpha a_0\right)t\right], g_2, g_3\right)\right\}$$

where

$$g_2 = -\frac{4a_0(\beta - 2\alpha a_0^2)}{\alpha a_1^3} \qquad g_3 = -\frac{4(\beta - 3\alpha a_0^2)}{\alpha a_1^2}.$$

Example 8. The Schrödinger–Boussinesq system

$$\begin{aligned} &\mathrm{i}u_t = u_{xx} + uv \\ &-v_{tt} + v_{xx} + (v^2)_{xx} - v_{xxxx} = (|u|^2)_{xx} \end{aligned} \tag{3.31}$$

is known to describe various physical processes in lasers and plasmas, such as formation, Langmuir field amplitude and intense electromagnetic waves and modulational instabilities [50–53]. The problem of the complete integrability of system (3.31) has been studied by Chowdhury *et al* from the view of Painlevé analysis [54]. The solitary wave solutions for system (3.31) were obtained in [54, 55]. Here our proposed method gives a series of travelling wave solutions as follows.

By considering transformations $u = e^{i\theta}U(\xi), v = V(\xi), \theta = px + qt, \xi = x + ct$, from system (3.31) we obtain the relation $c = 2p$ and coupled nonlinear ordinary differential equations

$$(q - p^2)U + UV + U'' = 0$$

$$(1 - 4p^2)V'' + (V^2)'' - (U^2)'' - V'''' = 0.$$

In a way similar to examples 6 and 7, we find that equation (3.31) admits a solitary wave solution

$$\begin{aligned} u_1 &= \pm 2\sqrt{2}c_2 e^{i\theta_1}[2 - 3\operatorname{sech}^2(\sqrt{c_2}\xi_1)] \\ v_1 &= -4c_2 + 6c_2\operatorname{sech}^2(\sqrt{c_2}\xi_1) \qquad c_2 > 0 \end{aligned} \tag{3.32}$$

a Jacobi doubly periodic solution

$$\begin{aligned} u_2 &= \mp 2\sqrt{2}c_2\, e^{i\theta_1}\left[2 - \frac{3m^2}{2m^2-1}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi_1\right)\right] \\ v_2 &= -4c_2 + \frac{6c_2m^2}{2m^2-1}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi_1\right) \qquad c_2 > 0 \end{aligned} \tag{3.33}$$

and a Weierstrass periodic solution

$$\begin{aligned} u_3 &= \pm\sqrt{2}\, e^{i\theta_2}[b_0 + b_1\wp(\sqrt{-b_1/6}\xi_2, g_2, g_3)] \\ v_3 &= b_0 + b_1 c\wp(\sqrt{-b_1/6}\xi_2, g_2, g_3) \qquad c_2 = 0 \qquad b_1 > 0 \end{aligned}$$

where

$$\begin{aligned} \xi_1 &= x \pm \sqrt{1 + 4c_2}t & \theta_1 &= \pm\sqrt{1 + 4c_2}x + \tfrac{1}{4}(1 + 20c_2)t \\ \xi_2 &= x \pm \sqrt{1 - 2b_0}t & \theta_2 &= \pm\sqrt{1 - 2b_0}x + \tfrac{1}{4}(1 - 10b_0)t \\ g_2 &= \frac{12b_0^2}{b_1^2} & g_3 &= -\frac{6c_0}{b_1}. \end{aligned}$$

As $m \to 1$, the Jacobi periodic solution (3.33) degenerates to the solitary wave solution (3.32).

4. Further discussion

Apart from the equations considered in this paper, the proposed method is also readily applicable to a large variety of other nonlinear equations including classical KdV, MdV, KdV–MKdV, Jaulent–Miodek, BBM, modified BBM, Benjamin Ono, Kawachra, Schrödinger, Klein–Gordon, sine-Gordon, sinh-Gordon, $(2+1)$-dimensional KP, $(2+1)$-dimensional Kaup–Kupershmidt, $(2+1)$-dimensional Gardner, coupled KdV, coupled Schrödinger-KdV and coupled Ito equations etc. According to the method, the travelling wave solutions of a given nonlinear equation depend on the explicit solvability of (2.5) with its coefficients c, a_i, c_j satisfying a system of algebraic equations. In the present paper, we have only investigated a special case when $r = 4$. The proposed method can be extended to the case when $r > 4$. In addition, we only consider travelling wave solutions involving a single quantity $x + ct$. We hope to further extend our proposed method to find multi-wave solutions. The details for these cases also will be investigated in our future works.

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References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Matveev V B and Salle M A 1991 *Darboux Transformation and Solitons* (Berlin: Springer)
[3] Gu C H, Hu H S and Zhou Z X 1999 *Darboux Transformations in Soliton Theory and its Geometric Applications* (Shanghai: Shanghai Science and Technology Publication)
[4] Esteevez P G 1999 *J. Math. Phys.* **40** 1406
[5] Dubrousky V G and Konopelchenko B G 1994 *J. Phys. A: Math. Gen.* **27** 4719
[6] Neugebauer G and Kramerl D 1983 Einstein–Maxwell solitons *J. Phys. A: Math. Gen.* **16** 1927
[7] Fan E G 2000 *J. Phys. A: Math. Gen.* **33** 6925
[8] Hirota R and Satsuma J 1981 *Phys. Lett.* A **85** 407
[9] Matsumno Y 1984 *Bilinear Transformation Method* (New York: Academic)
[10] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[11] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[12] Belokolos E, Bobenko A, Enol'skij V, Its A and Matveev V B 1994 *Algebro-Geometrical Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[13] Christiansen P L, Eilbeck J C, Enolskii V Z and Kostov N A 1995 *Proc. R. Soc.* A **451** 685
[14] Alber M S and Fedorov Y N 2001 *Inverse Problems* **17** 1017
[15] Novikov D P 1999 *Siberian Math. J.* **40** 136
[16] Malfliet W 1992 *Am. J. Phys.* **60** 650
[17] Hereman W 1996 *Comput. Phys. Commun.* **65** 143
[18] Parkes E J and Duffy B R 1996 *Comput. Phys. Commun.* **98** 288
[19] Parkes E J 1994 *J. Phys. A: Math. Gen.* **27** L497
[20] Fan E G 2000 *Phys. Lett.* A **277** 212
[21] Fan E G 2002 *Comput. Math. Appl.* **43** 671
[22] Yao Y X and Li Z B 2002 *Phys. Lett.* A **297** 196
[23] Li Z B, Liu Y P and Wang M L 2002 *Acta Math. Sin.* **22** 138
[24] Samsonov A M 1998 *Phys. Lett.* A **245** 527
[25] Porubov A V 1993 *J. Phys. A: Math. Gen.* **26** L797
[26] Porubov A V and Paeker D F 1999 *Wave Motion* **29** 97
[27] Akhiezer N L 1990 *Elements of Theory of Elliptic Functions* (Providence, RI: American Mathematical Society)
[28] Wang Z X and Xia X J 1989 *Special Functions* (Singapore: World Scientific)
[29] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Phys. Lett.* A **289** 69
[30] Liu S K, Fu Z T, Liu S D and Zhao Q 2002 *Acta Phys. Sin.* **51** 10
[31] Wilson G 1988 *Phys. Lett.* A **132** 445
[32] Calogero F and Degasperis A 1981 *J. Math. Phys.* **22** 23
[33] Yu S J and Toda K 2000 *J. Nonlinear Math. Phys.* **7** 1
[34] Jimbo M and Miwa T 1983 *Publ. Res. Inst. Math. Soc. Kyoto Univ.* **19** 943
[35] Dorrizzi B, Grammaticos B, Ramani A and Winternitz P 1986 *J. Math. Phys.* **27** 2848
[36] Hong W and Oh K S 1999 *Comput. Math. Appl.* **39** 29
[37] Seyler C E and Fenstermacher D L 1984 *Phys. Fluids* **27** 4
[38] Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 3821–48
[39] Drinfel'd V G and Sokolov V V 1981 *Sov. Math. Dokl.* **23** 457
[40] Wilson G 1982 *Phys. Lett.* A **89** 332
[41] Hirota R, Grammaticos B and Ramani A 1986 *J. Math. Phys.* **27** 1499
[42] Kupershmidt B A 1985 *Commun. Math. Phys.* **99** 51
[43] Boiti M, Leon J J P and Pempinelli F 1987 *Inverse Problems* **3** 371
[44] Goldobin E, Sterck A and Koelle D 2001 *Phys. Rev.* E **63** 03111
[45] Leung K M, Mills D L, Riseborough P S and Trullinger S E 1983 *Phys. Rev.* B **27** 4017

[46] Salerno M and Quintero N R 2002 *Phys. Rev.* E **65** 025602
[47] Lou S Y and Ni G J 1989 *Phys. Lett.* A **140** 33
[48] Gani V A and Kudryavtsev A E 1999 *Phys. Rev.* E **60** 3305
[49] Fokas A S 1980 *J. Math. Phys.* **21** 1318
[50] Rao N N and Shukla P K 1997 *Phys. Plasmas* **4** 636
[51] Shatashvili N L and Rao N N 1999 *Phys. Plasmas* **4** 66
[52] Saha P, Banerjee S and Roy Chowdhury A 2002 *Chaos Solitons Fractals* **14** 145
[53] Roy Chowdhury A, Dasgupta B and Rao N N 1997 *Chaos Solitons Fractals* **9** 1747–53
[54] Hase H and Satsuma J 1988 *J. Phys. Soc. Japan* **57** 679
[55] Conte R and Musette M 1992 *J. Phys. A: Math. Gen.* **25** 5609
[56] Senthilvelan M 2001 *Appl. Math. Comput.* **123** 381